UNITÉ DE RECHERCHE
INRIA-LORRAINE

Rapports de Recherche

1992
25ème anniversaire

N° 1624

Programme 5
Traitement du Signal,
Automatique et Productique

SUPPLY MANAGEMENT IN ASSEMBLY SYSTEMS : THE BASIC PROBLEM

Chengbin CHU
Jean-Marie PROTH
Xiaolan XIE

Février 1992

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

SUPPLY MANAGEMENT IN ASSEMBLY SYSTEMS: THE BASIC PROBLEM

Chengbin CHU (*), Jean-Marie PROTH (**) and Xiaolan XIE (*)

ABSTRACT:

We consider the case when n components are needed to assemble a given product. Components are provided by suppliers, and the period between the order time and the time a component is available (i.e. the yield from supplier) is the value taken by a random variable with a known distribution. The due date for the assembled product is also known. The costs to be taken into account are the inventory costs of the components, and the backlogging cost of the assembled product. We propose an iterative algorithm which leads to the optimal order instants of the components.

KEY WORDS:

Assembly process, Supply management, Random supply.

(*) INRIA-Lorraine, CESCOM, Technopôle Metz 2000, 4 rue Marconi, F-57070 METZ.

(**) INRIA-Lorraine, CESCOM, Technopôle Metz 2000, 4 rue Marconi, F-57070 METZ, and Systems Research Center, University of Maryland, College Park, MD 20742, USA.

GESTION DES COMPOSANTS DANS LES SYSTEMES D'ASSEMBLAGE : LE PROBLEME DE BASE

Chengbin CHU, Jean-Marie PROTH and Xiaolan XIE

RESUME :

Nous considérons le cas où n composants sont nécessaires pour assembler un produit donné. Les composants sont commandés chez des fournisseurs, et les temps de livraison sont des variables aléatoires quelconques connues. Le délai associé au produit fini est également connu. Les coûts à prendre en compte sont les coûts de stockage des composants et le coût de rupture du produit fini. Nous proposons un algorithme itératif qui conduit aux instants de commande optimaux des composants.

MOTS-CLE

Processus d'assemblage, Gestion des réapprovisionnements, Réapprovisionnements aléatoires.

1. INTRODUCTION

Large companies, such as car, computer or air plane builders, often order most of the components that they use from subcontractors who are in charge of providing these components on a just-in-time basis. This forces the suppliers to keep components in stock to face their production randomness. This, in turn, results in an increase of the component costs, and thus of the production costs of the final products. A new management policies for companies consists in managing themselves the production randomness of the components, provided that the prices of these components are lower than the usual ones. In other words, companies accept some randomness in the yields of subcontractors if the prices of the components are low, because they expect to take advantage, in real time, of using several suppliers.

In the literature, reported works essentially deal with supplier selection strategies (Cohen and Lee (1989), Hahn et al. (1986), Hendrick and Ruch (1988), Newman (1988) and Trevelen and Schweikhart (1988)). Haresh et al. (1990) address the supplier diversification when the quantity delivered is uncertain. Ranga et al. (1991) have considered the randomness of the lead time. However, they have only considered the inventory control problem of one material. In addition, they have only studied the cases where the distributions of the lead times are uniform or exponential.

In this paper, we assume that the suppliers have been selected a priori, on the basis of a supplier selection policy.

We restrict ourselves to the basic problem for which components of products to be assembled are ordered on demand. This is, in particular, the case when the products are manufactured to satisfy specific orders (air planes, large computers, some robots, manufacturing cells, etc...). We assume that the due date of assembled products is known, as well as the assembly time, which is deterministic. Thus, the instant at which the components should be available is equal to the due date minus the assembly time. Hereafter, this instant is referred to as the availability time.

The time between the ordering instant and the delivery instant of a component is the value taken by a continuous random variable, having a known distribution. Inventory costs of the components and backlogging cost of each assembled product are known. We are interested in finding the optimal ordering instants, i.e. the ordering instants of the components which minimize the total expected cost.

The problem is set in section 2. The properties related to this problem are given in section 3. An iterative algorithm is derived from these properties and presented in section 4. Finally, a numerical example is proposed to illustrate the algorithm in section 5.

2. PROBLEM FORMULATION

Let 0 be the availability time and n the number of components needed to assemble the final product.

For $i = 1, 2, ..., n$:

* $(-x_i)$ is the ordering instant of component i. The x_i's values have to be defined;

* l_i is the lead time of component i, i.e. the period between the ordering instant and the delivery instant of component i; l_i is a random variable the density probability of which is φ_i. ϕ_i denotes the related distribution function;

* α_i is the instantaneous inventory holding cost of component i.

The instantaneous backlogging cost is equal to one (otherwise, we have to divide all the instantaneous inventory holding costs by the instantaneous backlogging cost to comply with this hypothesis).

Later on, we will use the following notations:

$X = (x_1, x_2, ..., x_n)^T$

$L = (l_1, l_2, ..., l_n)^T$

$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$

where T denotes the matrix transpose.

We also use A as $\sum_{i=1}^{n} \alpha_i + 1$ (i.e. $A = \sum_{i=1}^{n} \alpha_i + 1$).

Using the previous notations, the total cost f (X, L) is given by:

$$f(X,L) = \alpha^T(X - L) + A \max_{i \in \{1,2,...,n\}} (l_i - x_i)^+ \tag{1}$$

where $s^+ = \max(0, s)$.

Thus, the total expected cost is:

$$\begin{aligned} C(X) &= E[f(X,L)] = \overline{f(X,L)} \\ &= \alpha^T(X - E[L]) + A.E\left[\max_{i \in \{1,2,...,n\}} (l_i - x_i)^+\right] \end{aligned} \tag{2}$$

or:

$$C(X) = \alpha^T(X - \overline{L}) + A.E\left[\max_{i \in \{1,2,...,n\}} (l_i - x_i)^+\right] \tag{3}$$

For k = 1, 2, ..., n, we define I_k (X, L) as follows:

$$I_k(X,L) = \begin{cases} 1 \text{ if } l_k - x_k = \max_{i \in \{1,2,...,n\}} (l_i - x_i)^+ \\ 0 \text{ otherwise} \end{cases} \tag{4}$$

In other words, I_k (X, L) is equal to 1 if the component k is the last component available and if it arrives after the availability time (i.e. after time 0).

Using definition (4), relation (3) can be rewritten as:

$$C(X) = \alpha^T(X - \overline{L}) + A \sum_{i=1}^{n} E\left[(l_i - x_i) I_i(X,L)\right] \tag{5}$$

(Note that the probability for I_i (X,L) = I_j (X,L) =1, $i \neq j$, is equal to 0 because the l_i random variables are continuous).

But:

$E[(l_i - x_i)\, I_i(X,L)]$

$$= \int_{y=0}^{+\infty} y \Pr\left\{l_k - x_k \le y, \forall k \ne i\right\} d\phi_i(x_i + y)$$

because the random variables l_i are independent from each other

or:

$E[(l_i - x_i)\, I_i(X,L)]$

$$= \int_{y=0}^{+\infty} y \prod_{k \ne i} \phi_k(x_k + y) \,.\, \varphi_i(x_i + y)\, dy \tag{6}$$

Finally, the problem consists in finding the components of X which minimize (see (5) and (6)):

$$C(X) = \alpha^T(X - \bar{L}) + A \sum_{i=1}^{n} \int_{y=0}^{+\infty} y \prod_{k \ne i} \phi_k(x_k + y) \,.\, \varphi_i(x_i + y)\, dy \tag{7}$$

$$= \alpha^T(X - \bar{L}) + A \int_{y=0}^{+\infty} \int_0^y \sum_{i=1}^{n} \prod_{k \ne i} \phi_k(x_k + y) \,.\, \varphi_i(x_i + y)\, dy\, dz$$

$$= \alpha^T(X - \bar{L}) + A \int_{z=0}^{+\infty} \int_{y=z}^{+\infty} \sum_{i=1}^{n} \prod_{k \ne i} \phi_k(x_k + y) \,.\, \varphi_i(x_i + y)\, dy\, dz$$

$$= \alpha^T(X - \bar{L}) + A \int_{y=0}^{+\infty} \left[1 - \prod_{k=1}^{n} \phi_k(x_k + y)\right] dy \tag{8}$$

3. TOTAL EXPECTED COST: PROPERTIES AND RESULTS

The following set of properties concerns the total expected cost function.

Properties 1:

We assume that: $\int_0^{\infty} x\varphi_i(x)\, dx$ exists for any $i \in \{1,2,\dots,n\}$ (i.e. the mean value of l_i exists) (9)

In this case, the total expected cost C(X) is continuous, convex and derivable.
Furthermore, the partial derivates can be written as:

$$\frac{\partial C(X)}{\partial x_k} = \alpha_k - A \int_0^{\infty} \prod_{i \ne k} \phi_i(x_i + y).\varphi_k(x_k + y) dy \text{ for } k = 1,2,\dots,n \tag{10}$$

Proof:

a. C(X) is continuous

Let $\Delta = (\Delta_1, \Delta_2, \dots, \Delta_n)^T$. According to relation (1):

$$|f(X+\Delta,L)-f(X,L)| \le \sum_{i=1}^{n} \alpha_i |\Delta_i| + A \underset{i\in\{1,2,\dots,n\}}{\mathcal{M}ax} |\Delta_i| \qquad (11)$$

According to Jensen's inequality and to relation (3):

$$\begin{aligned} |C(X+\Delta)-C(X)| &= |E[f(X+\Delta,L)]-E[f(X,L)]| \\ &\le E[|f(X+\Delta,L)-f(X,L)|] \end{aligned}$$

As a consequence (and taking into account inequality (11)):

$$|C(X+\Delta)-C(X)| \le \sum_{i=1}^{n} \alpha_i |\Delta_i| + A \underset{i\in\{1,2,\dots,n\}}{\mathcal{M}ax} |\Delta_i|$$

which tends to 0 when the components of vector Δ tend to 0.

Thus:

$\lim_{\Delta\to 0} C(X+\Delta) = C(X)$, which implies that C(X) is continuous.

b. C(X) is convex

$(l_i - x_i)^+$ is convex with regard to x_i for $i = 1,2,\dots,n$.

Thus, $\underset{i\in\{1,2,\dots,n\}}{\mathcal{M}ax} (l_i - x_i)^+$ is convex with regard to x_i and, as a consequence (see (1)), f(X,L) is convex with regard to X. Finally, $C(X) = E[f(X,L)]$ is convex with regard to X (i.e. with regard to the components of X).

c. C(X) is derivable with regard to any component of X

As shown in relation (8):

$$C(X) = \alpha^T (X - \overline{L}) + A. \int_{y=0}^{+\infty} \left[1 - \prod_{k=1}^{n} \phi_k(x_k + y)\right] dy$$

c1. We first prove that:

$$I_0 = \int_{y=0}^{+\infty} \frac{\partial}{\partial x_k} \left[1 - \prod_{k=1}^{n} \phi_k(x_k + y)\right] dy \text{ converges uniformly}$$

For any $a \ge 0$, we consider:

$$\begin{aligned} I_a &= \int_{y=a}^{+\infty} \frac{\partial}{\partial x_k} \left[1 - \prod_{k=1}^{n} \phi_k(x_k + y)\right] dy \\ &= -\int_{y=a}^{+\infty} \prod_{i\ne k} \phi_i(x_i + y).\varphi_k(x_k + y)\, dy \end{aligned}$$

Because $0 \le \phi_k(y) \le 1$ whatever $k \in \{1,2,\dots,n\}$ and $y \ge 0$, the following inequalities hold:

$$|I_a| \le \int_{y=a}^{+\infty} \varphi_k(x_k + y)\, dy = 1 - \phi_k(x_k + a) \le 1 - \phi_k(a)$$

Let us consider $1 > \varepsilon > 0$. We try to find a such that:

$$|I_a| \le \varepsilon$$

This inequality holds if:

$1 - \phi_k(a) \le \varepsilon$

$\phi_k(a) \le 1 - \varepsilon$

$a \ge \phi_k^{-1}(1-\varepsilon)$

Finally:

$\forall\, 1 > \varepsilon > 0, \exists\, M = \phi_k^{-1}(1-\varepsilon) > 0$ s.t.

$a \ge M \Rightarrow |I_a| \le \varepsilon$

Thus I_0 converges uniformly.

c2. We now prove that:

$$J = \int_{y=0}^{+\infty} \left[1 - \prod_{k=1}^{n} \phi_k(x_k + y)\right] dy \text{ converges}$$

Considering relation (7):

$$J = \sum_{i=1}^{n} \int_{y=0}^{+\infty} y \prod_{k \ne i} \phi_k(x_k + y).\varphi_i(x_i + y)dy$$

$$\le \sum_{i=1}^{n} \int_{y=0}^{+\infty} y.\varphi_i(x_i + y)dy \text{ because } 0 \le \prod_{k \ne i} \phi_k(x_k + y) \le 1$$

Thus, by setting $u = x_i + y$:

$$J \le \sum_{i=1}^{n} \left[\int_{u=x_i}^{+\infty} u.\varphi_i(u)du - x_i \int_{u=x_i}^{+\infty} \varphi_i(u)du\right]$$

$$J \le \sum_{i=1}^{n} E_i[l_i]$$

and thus J converges.

c3. As a consequence of c1 and c2,

$$\frac{\partial C(X)}{\partial x_k} = \alpha_k - A.\int_{y=0}^{+\infty} \prod_{i \ne k} \phi_i(x_i + y).\varphi_k(x_k + y)dy, \text{ for } k = 1,2,\ldots,n$$

Q.E.D.

C(X) being a convex function, C(X) is minimal for $X^* = (x_1^*, \ldots, x_n^*)$ if and only if:

$$\frac{\partial C}{\partial x_k}(X^*) = 0 \text{ for } k = 1,2,\ldots,n$$

or, according to (18):

$$\alpha_k - A\int_0^{\infty} \prod_{i \ne k} \phi_i(x_i^* + y)\varphi_k(x_k^* + y)dy = 0 \text{ for } k = 1,2,\ldots,n$$

which can be rewritten as:

$$\int_0^{\infty} \prod_{i \ne k} \phi_i(x_i^* + y)\, \varphi_k(x_k^* + y)dy = \alpha_k / A, \text{ for } k = 1,2,\ldots,n \tag{12}$$

We propose a necessary condition for X^* to be optimal (i.e. to be such that $C(X^*) = \min_X C(X)$).

Result 1:

For any optimal solution X*, the following conditions hold:

a. $\prod_{i=1}^{n} \phi_i(x_i^*) = 1/A$

b. $1/A \le \phi_i(x_i^*) \le 1 - \alpha_i / A$, for $i = 1,2,\dots,n$

c. $0 \le x_i^* < +\infty$, for $i = 1,2,\dots,n$

Proof:

a. From relation (12), we derive:

$$\sum_{k=1}^{n} \int_0^\infty \prod_{i \ne k} \phi_i(x_i^* + y)\, \varphi_k(x_k^* + y) dy = \sum_{k=1}^{n} \alpha_k / A$$

$$\int_0^\infty \sum_{k=1}^{n} \prod_{i \ne k} \phi_i(x_i^* + y)\, \varphi_k(x_k^* + y) dy = \sum_{k=1}^{n} \alpha_k / A$$

which can be rewritten as:

$$\int_0^\infty \frac{\partial}{\partial y}\left[\prod_{i=1}^{n} \phi_i(x_i^* + y)\right] dy = \sum_{k=1}^{n} \alpha_k / A$$

or:

$$\left[\prod_{i=1}^{n} \phi_i(x_i^* + y)\right]_0^\infty = \sum_{k=1}^{n} \alpha_k / A$$

$$1 - \prod_{i=1}^{n} \phi_i(x_i^*) = \sum_{k=1}^{n} \alpha_k / A$$

and:

$$\prod_{i=1}^{n} \phi_i(x_i^*) = 1 - \sum_{k=1}^{n} \alpha_k / A = 1/A \text{ (because } A = 1 + \sum_{k=1}^{n} \alpha_k) \tag{13}$$

b. From relation (13), and taking into account the fact that $\phi_i(x) \le 1, \forall x \in [0, +\infty), \forall i \in \{1,2,\dots,n\}$:

$$\phi_i(x_i^*) \ge 1/A \text{ for } i = 1,2,\dots,n \tag{14}$$

Furthermore, because $\prod_{i \ne k} \phi_i(x_i^* + y) \le 1$, relation (12) leads to:

$$\int_0^\infty \varphi_k(x_k^* + y) dy \ge \alpha_k / A, \text{ for } k = 1,2,\dots,n$$

Thus:

$$\left[\phi_k(x_k^* + y)\right]_0^\infty \ge \alpha_k / A$$

$$1 - \phi_k(x_k^*) \ge \alpha_k / A$$

and, finally:

$$\phi_k(x_k^*) \le 1 - \alpha_k / A, \text{ for } k = 1,2,\dots,n \tag{15}$$

And, from relations (14) and (15):

$$1/A \le \phi_i(x_i^*) \le 1 - \alpha_i / A, \text{ for } k = 1,2,\dots,n$$

c. The last result means that all the components are ordered at the latest at the delivery instant. It is the consequence of the fact that $\phi_i(x_i^*) \geq 1/A > 0$ whatever $i \in \{1,2,...,n\}$ (see relation (14)) and that $\phi_i(x) = 0$ for $x < 0$ (because the lead time of any component is greater than or equal to zero).

Q.E.D.

The next result connects the variations of the distribution functions with the current values of the partial derivates.

Let Δ_k be the vector of dimension n which components are equal to 0, except the k-th one which is equal to $\Delta > 0$.

Result 2:

The following results hold:

a. If $\dfrac{\partial C(X)}{\partial x_k} < 0$

and:

$$A\left[\phi_k(y+\Delta) - \phi_k(y)\right] \leq -\frac{\partial C(X)}{\partial x_k}, \forall\ y \geq x_k$$

then:

$$C\ (X + \Delta_k) < C(X),$$

$$\frac{\partial C(X+\Delta_k)}{\partial x_k} \leq 0, \text{ and } \frac{\partial C(X+\Delta_k)}{\partial x_i} \leq \frac{\partial C(X)}{\partial x_i}, \forall i \in \{1,2,\ldots,n\}, i \neq k$$

b. If $\dfrac{\partial C(X)}{\partial x_k} > 0$

and:

$$A\left[\phi_k(y) - \phi_k(y-\Delta)\right] \leq \frac{\partial C(X)}{\partial x_k}, \forall\ y \geq x_k$$

then:

$$C\ (X - \Delta_k) < C(X),$$

$$\frac{\partial C(X-\Delta_k)}{\partial x_k} \geq 0, \text{ and } \frac{\partial C(X-\Delta_k)}{\partial x_i} \geq \frac{\partial C(X)}{\partial x_i}, \forall i \in \{1,2,\ldots,n\}, i \neq k$$

Proof:

Considering result 1, we can derive the following formulation of the partial derivate of C with regard to x_k, the l_i's values being fixed for $i \neq k$:

$$\frac{\partial C(X)}{\partial x_k} = \alpha_k - A\ \Pr\left\{l_k - x_k \geq \max_{i \neq k}(l_i - x_i)^+\right\} \tag{16}$$

$$= \alpha_k - A\ \Pr\left\{l_k \geq x_k + \max_{i \neq k}(l_i - x_i)^+\right\}$$

$$= \alpha_k - A\left(1 - E\left[\phi_k\left(x_k + \max_{i \neq k}(l_i - x_i)^+\right)\right]\right) \tag{17}$$

a. Case $\dfrac{\partial C(X)}{\partial x_k} < 0$

From relation (17), we derive:

$$\frac{\partial C(X+\Delta_k)}{\partial x_k} - \frac{\partial C(X)}{\partial x_k}$$

$$= A.E\left[\phi_k\left(x_k + \Delta + \max_{i \neq k}(l_i - x_i)^+\right) - \phi_k\left(x_k + \max_{i \neq k}(l_i - x_i)^+\right)\right] \tag{18}$$

But, according to the second hypothesis of result 2, a.:

$$A.E\left[\phi_k\left(x_k + \Delta + \max_{i \neq k}(l_i - x_i)^+\right) - \phi_k\left(x_k + \max_{i \neq k}(l_i - x_i)^+\right)\right] \leq -\frac{\partial C(X)}{\partial x_k} \tag{19}$$

Thus, from (18) and (19) we derive:

$$\frac{\partial C(X+\Delta_k)}{\partial x_k} \leq 0$$

and, because $\dfrac{\partial C(X)}{\partial x_k} < 0$ and $\Delta > 0$,

$$\frac{\partial C(X+\Delta_k)}{\partial x_k} \leq C(X)$$

From relation (16), we derive:

$$\frac{\partial C(X)}{\partial x_i} = \alpha_i - A.\Pr\left\{l_i - x_i \geq \max\left\{\max_{m \neq k; m \neq i}(l_m - x_m)^+, (l_k - x_k)^+\right\}\right\} \tag{20}$$

Thus:

$$\frac{\partial C(X+\Delta_k)}{\partial x_i} = \alpha_i - A.\Pr\left\{l_i - x_i \geq \max\left\{\max_{m \neq k; m \neq i}(l_m - x_m)^+, (l_k - x_k - \Delta)^+\right\}\right\}$$

$$\leq \alpha_i - A.\Pr\left\{l_i - x_i \geq \max\left\{\max_{m \neq k; m \neq i}(l_m - x_m)^+, (l_k - x_k)^+\right\}\right\} \tag{21}$$

And, from relations (20) and (21), it follows that:

$$\frac{\partial C(X+\Delta_k)}{\partial x_i} \leq \frac{\partial C(X)}{\partial x_i}$$

b. Case $\dfrac{\partial C(X)}{\partial x_k} > 0$

The proof is similar to the previous one.

Q.E.D.

A consequence of the second result is given by corollary 1.

Corollary 1:

If the probability densities are upper bounded, i.e.:

$$\varphi_k(x_k) \leq m_k, \ \forall\ k = 1,2,\ldots,n$$

then:

a. If $\frac{\partial C(X)}{\partial x_k} < 0$

and:

$$\Delta = -\frac{\partial C(X)}{\partial x_k} / (A.m_k)$$

then:

$$C(X + \Delta_k) < C(X)$$

b. If $\frac{\partial C(X)}{\partial x_k} > 0$

and:

$$\Delta = \frac{\partial C(X)}{\partial x_k} / (A.m_k)$$

then:

$$C(X - \Delta_k) < C(X)$$

Note that the probability densities are always bounded in the case of real life problems.

Result 3

The following two results are the basis of the algorithm presented in the next section.

a. We consider:

$$X = (x_1, x_2, \ldots, x_n),\ x_i \geq 0,\ \forall i \in \{1,2,\ldots,n\}$$

and define:

$$E_X^- = \left\{ k / k \in \{1,2,\ldots,n\} \text{ and } \frac{\partial C(X)}{\partial x_k} \leq 0 \right\}$$

We then construct $X' = \{x_1', x_2', \ldots, x_n'\}$ as follows:

$$\begin{cases} x_k' = x_k \text{ if } k \in E_X^- \\ 0 \leq x_k' \leq x_k \text{ and } x_k' \text{ is such that } A.\left[\phi_k(y + x_k) - \phi_k(y + x_k')\right] \leq \frac{\partial C(X)}{\partial x_k}, \text{ for any } y \geq 0 \\ \qquad\qquad \text{if } k \notin E_X^- \end{cases} \tag{22}$$

The following inequalities hold:

$$C(X') < C(X) \tag{23}$$

and:

$$\frac{\partial C(X')}{\partial x_k} \geq 0 \text{ for every k such that } \frac{\partial C(X)}{\partial x_k} \geq 0 \tag{24}$$

b. We define:

$$F_X^+ = \left\{ k / k \in \{1,2,\ldots,n\} \text{ and } \frac{\partial C(X)}{\partial x_k} \geq 0 \right\}$$

We now construct $X'' = \{x_1'', x_2'', \ldots, x_n''\}$ as follows:

$$\begin{cases} x_k'' = x_k \text{ if } k \in F_X^+ \\ x_k'' \geq x_k \text{ and} x_k'' \text{ is such that } A.\left[\phi_k(y + x_k'') - \phi_k(y + x)\right] \leq \dfrac{\partial C(X)}{\partial x_k}, \text{ for any } y \geq 0 \\ \qquad\qquad \text{if } k \notin F_X^+ \end{cases} \tag{25}$$

In this case, the following inequalities hold:

$$C(X'') < C(X) \tag{26}$$

and:

$$\frac{\partial C(X'')}{\partial x_k} \leq 0 \text{ for every k such that } \frac{\partial C(X)}{\partial x_k} \leq 0 \tag{27}$$

Proof

We restrict ourselves to the proof of part a. The proof of part b is similar.

We define the sequence $X^0, X^1, ..., X^n$, where $X^k = \left(x_1^k, x_2^k, ..., x_n^k\right)$ for $k = 0,1,...,n$, as follows:

$$x_i^k = \begin{cases} x_i' \text{ if } i \leq k \\ x_i \text{ otherwise} \end{cases}$$

Of course, $X^0 = X$ and $X^n = X'$

According to result 2 and the definition of X^k we have, for $k = 1,2,...,n$ and in the case where $\dfrac{\partial C(X^{k-1})}{\partial x_k} > 0$:

$$C(X^k) < C(X^{k-1}) \tag{28}$$

$$\frac{\partial C(X^k)}{\partial x_k} \geq 0 \tag{29}$$

and:

$$\frac{\partial C(X^k)}{\partial x_i} \geq \frac{\partial C(X^{k-1})}{\partial x_i}, \forall i \in \{1,2,...,n\}, i \neq k \tag{30}$$

Considering relations (29) and (30), it is obvious that:

$$\frac{\partial C(X^0)}{\partial x_k} = \frac{\partial C(X)}{\partial x_k} \geq 0 \text{ leads to } \frac{\partial C(X^k)}{\partial x_k} \geq 0, \text{ for } k = 1,2,...,n$$

and thus:

$$\frac{\partial C(X')}{\partial x_k} \geq 0$$

Q.E.D.

4. AN ITERATIVE ALGORITHM

The algorithm presented in this section starts from a vector X, the components of which are all positive, and at the maximal value possible for the optimal solution. As a consequence, all the partial derivates are positive. At each step of the computation, the components x_k of X are reduced in such a way that the partial derivates remain non negative.

Algorithm:

1. Set m: = 0
2. Initial solution:

   We set $x_k^m = \phi_k^{-1}\left(1 - \frac{\alpha_k}{A}\right)$ for $k = 1,2,\ldots,n$ (31)

   where $X^m = \left(x_1^m, x_2^m, \ldots, x_n^m\right)$

   The initial values are the upper bounds of the components of the optimal solution (see item b. of result 1).
3. Compute:

   $C(X^m)$ and $\frac{\partial C(X^m)}{\partial x_k}$ for $k = 1,2,\ldots,n$

   If no analytical formulation exists for the cost C, the value $C(X^m)$ can be evaluated by simulation using relation (3). The value of $\frac{\partial C(X^m)}{\partial x_k}$ can also be evaluated by simulation using relation (16).
4. Computation of the next solution

   For $k = 1,2,\ldots,n$:

   $$x_k^{m+1} = x_k^m - \left(\frac{\partial C(X^m)}{\partial x_k}\right) \Big/ \left(A.\sup_{y \geq r_k} \varphi_k(y)\right) \quad (32)$$

   where: $r_k = \phi_k^{-1}(1/A)$

   Note that r_k is a lower bound of the k-th component of the optimal solution.
5. If $\max\limits_{k \in \{1,2,\ldots,n\}} \frac{\partial C(X^{m+1})}{\partial x_k} \geq \varepsilon$, go to 3,

   Otherwise, stop the computation and keep X^{m+1} as the solution of the problem (ε is a "small" positive value provided by the user).

Note: Other tests can be used to stop the computation.

Result 4

The series $\{X^m\}_{m=1,2,...}$ converges to the optimal solution X^* of the problem (i.e. converges to X^* such that $\frac{\partial C(X^*)}{\partial x_k} = 0$ for $k = 1,2,...,n$).

Proof

The components $x_1^0, x_2^0, ..., x_n^0$ of the initial solution (see (31)) are the upper bounds of the components of the optimal solution (see item b. of result 1).

As a consequence, $\frac{\partial C(X^0)}{\partial x_k} \geq 0$ for every k $\{1,2,...,n\}$.

If $\frac{\partial C(X^0)}{\partial x_k} > 0$, we are in the case a. of result 3 where X (resp. X') is replaced by X^0 (resp. X^1).

$X^1 \leq X^0$ must be chosen in such a way that (see (22)):

$$A.\left[\phi_k\left(y + x_k^0\right) - \phi_k\left(y + x_k^1\right)\right] \leq \frac{\partial C(X^0)}{\partial x_k} \text{ for } k = 1,2,\ldots,n \tag{33}$$

Note that inequality (33) holds if:

$$A.\left[x_k^0 - x_k^1\right] \sup_{y \geq r_k} \varphi_k(y) \leq \frac{\partial C(X^0)}{\partial x_k} \text{ for } k = 1,2,\ldots,n \tag{34}$$

where $r_k = \phi_k^{-1}(1/A)$, which is the minimal value of the k-th components of X^*, i.e. x_k^*.

We derive from (34):

$$x_k^1 = x_k^0 - \left[\frac{\partial C(X^0)}{\partial x_k}\right] \Big/ \left[A. \sup_{y \geq r_k} \varphi_k(y)\right] \text{ for } k = 1,2,\ldots,n \tag{35}$$

Note that relation (35) still holds if $\frac{\partial C(X^0)}{\partial x_k} = 0$. Thus, this relation applies for $\frac{\partial C(X^0)}{\partial x_k} \geq 0$.

According to result 3:

$$C(X^1) \leq C(X^0) \tag{36}$$

and:

$$\frac{\partial C(X^1)}{\partial x_k} \geq 0 \text{ for } k = 1,2,\ldots,n \tag{37}$$

The equality holds in relation (36) iff the equality holds for all the relations (37), and in that case $X^* = X^1 = X^0$ is optimal. Otherwise, we can construct X^2 starting from X^1 like X^1 has been derived from X^0, and so on until relation (36) and (37) become equalities: at this point, the solution is optimal (i.e. is X^*).

Q.E.D.

5. A NUMERICAL EXAMPLE

We illustrate the previous approach using an example where the final product is obtained from the assembly of two components.

For the first component:

the inventory cost is $\alpha_1 = 0.2$

the probability density of the lead time l_1 is:

$\varphi_1(l_1) = \exp(-l_1)$ for $l_1 \geq 0$ and 0 otherwise

For the second component:

the inventory cost is $\alpha_2 = 0.7$

the probability density of the lead time l_2 is:

$$\varphi_2(l_2) = \begin{cases} 1 & \text{if} \quad l_2 \in [4,5] \\ 0 & \text{otherwise} \end{cases}$$

As we can see, $m_1 = 1$ and $m_2 = 1$

Furthermore, $A = 1 + 0.2 + 0.7 = 1.9$

According to relation (7):

$$C(X) = [0.2, 0.7]\begin{bmatrix} x_1 - 1 \\ x_2 - 4.5 \end{bmatrix} + 1.9\left[\int_0^{\infty} y\phi_2(x_2 + y).\varphi_1(x_1 + y)dy + \int_0^{\infty} y\phi_1(x_1 + y).\varphi_2(x_2 + y)dy\right]$$

$$= 0.2x_1 + 0.7x_2 - 3.35 + 0.9 + \left[\int_{(4-x_2)^+}^{(5-x_2)^+} y(x_2 + y - 4)\exp(-x_1 - y)dy + \int_{(5-x_2)^+}^{+\infty} y\exp(-x_1 - y)dy\right.$$

$$\left. + \int_{(4-x_2)^+}^{(5-x_2)^+} y(1 - \exp(-x_1 - y))dy \right]$$

We obtain:

$$C(X) = 1.9\exp(x_2 - x_1)(\exp(-4) - \exp(-5)) + 5.2 - 1.2x_2 + 0.2x_1, \quad \text{if } x_2 \leq 4$$

$$C(X) = 0.2x_1 - 8.8x_2 + 20.4 + 0.95x_2^2 - 1.9\left[\exp(-x_1 + x_2 - 5) - (x_2 - 3)\exp(-x_1)\right] \text{if } 4 \leq x_2 \leq 5$$

$$C(X) = 1.9\exp(-x_1) - 3.35 + 0.7x_2 + 0.2x_1 \quad \text{if } x_2 \geq 5$$

We apply the above algorithm.

We first compute the partial derivates.

* If $x_2 \leq 4$

$$\frac{\partial C}{\partial x_1} = 0.2 - 1.9\left(\exp(-4) - \exp(-5)\right)\exp(x_2 - x_1)$$

$$\frac{\partial C}{\partial x_2} = -1.2 + 1.9\left(\exp(-4) - \exp(-5)\right)\exp(x_2 - x_1)$$

* If $4 \leq x_2 \leq 5$

$$\frac{\partial C}{\partial x_1} = 0.2 + 1.9\left[\exp(-x_1 + x_2 - 5) - (x_2 - 3)\exp(-x_1)\right]$$

$$\frac{\partial C}{\partial x_2} = -8.8 + 1.9x_2 + 1.9\left[\exp(-x_1 + x_2 - 5) - \exp(-x_1)\right]$$

* If $x_2 \geq 5$

$$\frac{\partial C}{\partial x_1} = 0.2 - 1.9\exp(-x_1)$$

$$\frac{\partial C}{\partial x_2} = 0.7$$

In this example, it can easily be seen that the optimal solution (x_1^*, x_2^*) is such that $x_2^* \in (4,5)$ by considering $\frac{\partial C}{\partial x_1} + \frac{\partial C}{\partial x_2}$, which must be equal to zero.

The sequence of solutions of partial derivates and of C(X) values obtained by applying the above algorithm are given in table 1 (we chose $\varepsilon = 10^{-5}$).

We can see that: $C(X^*) = 0.657641$

and $x_1^* = 2.176140 \quad x_2^* = 4.593694$

Table 1: Numerical application

Step	x_1	x_2	$\partial C/x_1$	$\partial C/x_2$	C
0	2.251292	4.631579	0.012049	0.061635	0.659262
1	2.191045	4.599140	0.002578	0.008518	0.657683
2	2.178909	4.594656	0.000487	0.001501	0.657642
3	2.176643	4.593866	0.000091	0.000276	0.657641
4	2.176219	4.593721	0.000017	0.000051	0.657641
5	2.176140	4.593694	0.000003	0.000010	0.657641

6. CONCLUSION

The algorithm proposed in this paper is particularly efficient when it is impossible to obtain an analytical value of the partial derivates. In this case, the partial derivates are computed at each step by simulation using the first right member of relation (16), whereas the objective function C (X) can be computed using relation (3).

The choice of the initial value of the solution has been made in order to keep positive partial derivates at each step of the computation. As a consequence, all the components of the solution are changed at each step of the computation. This explains the quick convergence of the algorithm.

The above presentation is based on the assumption that the l_k random variables are continuous and have bounded probability densities.

Further research work in the assembly systems area will deal with components which can be ordered from different suppliers who are more or less efficient in terms of lead time, the most efficient one being the most costly.

BIBLIOGRAPHY

[1] Cohen M. and Lee H., "Resource development analysis of global manufacturing and distribution networks", *Journal of Manufacturing and Operations Management*, 2.2, pp. 81-104 (1989).

[2] Feller W., *An introduction of probability theory and its applications*, Vol. I, Third edition, John Wiley (1968).

[3] Gurnani H.B., Akella R. and Lehoczky J., "Supply management in assembly systems - General yield model", *Working Paper*, Graduate School of Industrial Administration, Carnegie Mellon University (April 1990).

[4] Hahn C.K., Kim K.H. and Kim J.S., "Costs of competition: implications for purchasing strategy", *Journal of Purchasing and Materials Management*, 22.3, pp. 2-7 (1986).

[5] Hendrick T.E. and Ruch W.A., "Determining performance appraisal criterial for the buyer", *Journal of Purchasing and Materials Management*, 24.2, pp. 18-26 (1988).

[6] Newman R.G., "Single source qualification", *Journal of Purchasing and Materials Management*, 24.2, pp. 10-17 (1988).

[7] Ramasesh R.V., Ord J.K., Hayya J.C. and Pan A., "Sole versus dual sourcing i stochastic lead-time (s, Q) inventory models", *Management Science*, 37.4, pp. 428-443 (1991).

[8] Treleven M. and Schweikhart S.B., "A risk/benefit analysis of sourcing strategies: single vs. multiple sourcing", *Journal of Operations Management*, 7.4, pp. 93-114 (1988).

ISSN 0249-6399